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A sensitivity analysis of a class of semi-coercive variational inequalities using recession tools

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Abstract Using the recession analysis we study necessary and sufficient conditions for the existence and the stability of a finite semi-coercive variational inequality with respect to data perturbation. Some applications of the abstract results in mechanics and in electronic circuits involving devices like ideal diode and practical diode are discussed.

Keywords Finite variational inequalities · Recession analysis · Convex analysis · Positive semidefinite matrices · Clipping circuit · Ideal diode model · Practical diode model · coulomb friction

1 Introduction and position of the problem

The theory of variational inequalities, with its wide range of applications in engineering, economics, finance, industy and mechanics, has become a well-established and fruitful area of research. After the fundamental work of Lions and Stampacchia [18], this theory have been studied intensively. With the contributions of Brézis [7,8], Duvaut Lions [10], Browder [9],

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Kinderlehrer and Stampacchia [16], Panagiotopoulos [19], Goeleven Motreanu [13] (among others), this field has known an increasing growth in both theory and applications. Several books and articles have documented the basic theory, the numerical approach and applications in applied science as well. This theory was used as a tool for the study of partial differential equations with applications essentially drawn from mechanics (Signorini problem, obstacle problems in elasticity, etc).

We consider the following finite dimensional variational inequality

$$\mathrm{VI}(M, q, \Phi, K) \begin{cases} \text{Find } u \in K \text{ such that} \\ \langle Mu + q, v - u \rangle + \Phi(v) - \Phi(u) \geq 0, \quad \forall v \in K \end{cases}$$

where $M \in \mathbf{R}^{n\times n}$ is a matrix, $q \in \mathbf{R}^n$ is a vector, K is a nonempty closed convex set of $\mathbb{R}^n$ and $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous function.

We denote by:

$$\mathrm{Dom}\,(\Phi) = \{v \in \mathbb{R}^n \ : \ \Phi(v) < +\infty\},$$

the effective domain of Φ.

Let us now suppose that the assumptions $(\mathcal{H})$ described below are satisfied:

(H_1) $M \in \mathbb{R}^{n\times n}$ *is a symmetric and positive semidefinite matrix*;
(H_2) $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is a proper, convex, lower semicontinuous and bounded from below*;
(H_3) $K \subset \mathbb{R}^n$ *is a closed and nonempty convex set*
(H_4) $0 \in \mathrm{Dom}\ \Phi \cap K$.

Several theoretical existence results for $\mathrm{VI}(M, q, \Phi, K)$ in general reflexive Banach spaces and governed by a general operator M (not necessarily linear) are well known when a coerciveness condition hold for the operator M. We can cite for instance the contributions of Lions [17], Brézis [7,8], Browder [9] etc .... However, the variational formulation of many engineering problems leads generally to non-coercive variational inequalities (e.g., problems in mechanics which admits nontrivial virtual rigid body displacement). These problems are formulated by semi-coercive variational inequalities and was studied first by Fichera [12] and Lions and Stampacchia [18], Duvaut and Lions [10] (for problems with frictional type functionals). Recently many mathematicians and engineers has focused their attention on non-coercive unilateral problems, using several different approaches such as the critical point theory, the Leray-Schauder degree theory, the recession analysis or the regularization method by approximating non-coercive problems by coercive ones (see e.g., [1,4–6,21,22] and references cited therein). The main concern of these contributions is the obtainment of necessary or sufficient conditions for the solvability of such problems in a general setting by imposing some compacity conditions and some compatibility conditions on the right hand term q. More recently, Adly et al. [2,3] has considered the situations in which the existence of the solution is stable with respect to small uniform perturbations of the data of the problem. This result should be of great interest for problems in finance and engineering where the data are known only with a certain precision and it is desired that further refinement of the data should not cause the emptiness of the set of solutions.

Our aim is thus to characterize the sensitivity of $\mathrm{VI}(M, q, \Phi, K)$ with respect to the perturbations of the data M, q, Φ and K. In this paper, we only discuss the case of finite variational inequalities. Some applications of our main results in electronics and mechanics are given in Sect. 4.

2 Preliminaries and notations

Let us first recall some background results from convex analysis which will be used later.

Let K be a closed convex subset of $\mathbb{R}^n$, the *recession cone* of K is the closed convex cone

$$K_\infty := \bigcap_{t>0} \left[\frac{K - x_0}{t} \right],$$

with x_0 arbitrarily chosen in K.

We denote by $\Gamma_0(\mathbb{R}^n)$ the set of all proper, convex and lower semicontinuous extended real valued functions $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

Let $\Phi \in \Gamma_0(\mathbb{R}^n)$ be given. The recession function Φ_∞ of Φ is defined by:

$$\Phi_\infty(x) := \lim_{\lambda \to +\infty} \frac{\Phi(x_0 + \lambda x) - \Phi(x_0)}{\lambda}, \tag{1}$$

where $x_0 \in \text{Dom } \Phi$ is an arbitrary element. We set

$$\ker \Phi_\infty = \{x \in \mathbb{R}^n \; : \; \Phi_\infty(x) = 0\},$$

which is clearly a closed convex cone in $\mathbb{R}^n$.

The Fenchel conjugate $\Phi^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of Φ is defined by:

$$\Phi^*(x^*) = \sup_{x \in \mathbb{R}^n} \left\{ \langle x^*, x \rangle - \Phi(x) \right\}.$$

We recall also that the convex subdifferential of Φ at a point $x \in \text{Dom } \Phi$ is defined by:

$$\partial\Phi(x) = \{w \in \mathbb{R}^n \; : \; \langle w, y - x \rangle \leq \Phi(y) - \Phi(x), \quad \forall y \in \mathbb{R}^n\}.$$

The effective domain of the multivalued mapping $\partial\Phi$ is defined by

$$\text{D}(\partial\Phi) = \{x \in \mathbb{R}^n \; : \; \partial\Phi(x) \neq \emptyset\}.$$

We note that

$$\text{D}(\partial\Phi) \subset \text{ Dom } \Phi.$$

The range of the multivalued mapping $\partial\Phi$ is defined by

$$R(\partial\Phi) = \bigcup_{x \in \mathbb{R}^n} \partial\Phi(x).$$

We have the following Fenchel correspondence:

$$w \in \partial\Phi(x) \Longleftrightarrow x \in \partial\Phi^*(w).$$

Hence,

$$R(\partial\Phi) = D(\partial\Phi^*) \subset \text{Dom } \Phi^*. \tag{2}$$

The indicator function to a convex set K is given by:

$$I_K(x) = \begin{cases} 0 & \text{if } \; x \in K \\ +\infty & \text{if } \; x \notin K. \end{cases}$$

If K is a closed cone, its polar is defined by

$$K^\circ = \{x^* \in \mathbb{R}^n \; : \; \langle x^*, x \rangle \leq 0, \;\; \forall x \in K\}.$$

The support function to K is defined by:

$$\sigma_K(w) = (I_K)^*(w) = \sup_{x \in K} \langle w, x \rangle.$$

We recall that the barrier cone of K is defined by:

$$\mathcal{B}(K) = \{w \in \mathbb{R}^n \ : \ \sup_{x \in K} \langle w, x \rangle < +\infty\} = \text{Dom } \sigma_K. \tag{3}$$

It is well-known that if K is a non-empty closed and convex subset, then

$$\mathcal{B}(K)^\circ = K_\infty. \tag{4}$$

Therefore,

$$\overline{\mathcal{B}(K)} = (K_\infty)^\circ. \tag{5}$$

We recall that for $\Phi_1, \ \Phi_2 \in \Gamma_0(\mathbb{R}^n)$, the infimal convolution (or the epigraphical sum) is defined by:

$$(\Phi_1 \Box \Phi_2)(x) = \inf_{y+z=x} \{\Phi_1(y) + \Phi_2(z)\}. \tag{6}$$

We say that the infimal convolution is exact provided that the infimum appearing in (6) is achieved.
We note that

$$\text{Dom } (\Phi_1 \Box \Phi_2) = \text{Dom } (\Phi_1) + \text{Dom } (\Phi_2). \tag{7}$$

We recall that if Φ_1 (or Φ_2) is continuous on $\mathbb{R}^n$, then

$$(\Phi_1 + \Phi_2)^* = \Phi_1^* \Box \Phi_2^* \tag{8}$$

and the infimal convolution $\Phi_1^* \Box \Phi_2^*$ is exact.

Let us finally recall the following proposition

Proposition 1 *Let $\Psi \in \Gamma_0(\mathbb{R}^n)$ and $p \in \mathbb{R}^n$ be given. We have:*

(i) $p \in \overline{\text{Dom } \Psi^*} \iff \Psi_\infty(w) \geq \langle p, w \rangle, \quad \forall w \in \mathbb{R}^n$;
(ii) $p \in \text{Int } (\text{Dom } \Psi^*) \iff \Psi_\infty(w) > \langle p, w \rangle, \quad \forall w \in \mathbb{R}^n \setminus \{0\}$.

Proof For a proof see Corollary 13.3.4 in [20]. □

3 Characterization results

The solutions set of VI(M, q, Φ, K) will be denoted by

$$\text{Sol}(M, q, \Phi, K) := \{u \in K : \langle Mu + q, v - u \rangle + \Phi(v) - \Phi(u) \geq 0, \quad \forall v \in K\}.$$

The following resolvent set will also play an important role

$$\mathcal{R}(M, \Phi, K) = \{-q \in \mathbb{R}^n \ : \ \text{Sol}(M, q, \Phi, K) \neq \emptyset\}.$$

Let us introduce the following function $\Psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\Psi(u) = \frac{1}{2} \|Qu\|^2 + \Phi(u) + I_K(u), \tag{9}$$

where $Q = I - P_{\ker(M)}$ and $P_{\ker(M)}$ denotes the orthogonal projector from $\mathbb{R}^n$ to $\ker(M)$.

Remark 1 We first remark that $\mathrm{VI}(M, q, \Phi, K)$ is equivalent to the following variational inclusion: find $u \in K$ such that

$$0 \in Mu + q + \partial(\Phi + I_K)(u),$$

where I_K denotes the indicator function of K. Hence,

$$-q \in Mu + \partial(\Phi + I_K)(u) \subset \bigcup_{u \in \mathbb{R}^n} (Mu + \partial(\Phi + I_K)(u)) \subset R(M) + R(\partial(\Phi + I_K)).$$

Using (2), we get

$$-q \in R(M) + \mathrm{Dom}\,(\Phi + I_K)^*. \tag{10}$$

We have the following lemma.

Lemma 1 *Suppose that the assumptions* $(\mathcal{H})$ *hold. We have*

$$\mathrm{Dom}\,\Psi^* = R(M) + \mathrm{Dom}\,(\Phi + I_K)^*.$$

Proof Let us note first that the function Ψ defined in (9) can be rewritten

$$\Psi(u) = \frac{1}{2}\,(\mathrm{dist}_{\ker(M)}(u))^2 + (\Phi + I_K)(u).$$

Using (8) for $\Phi_1 = \frac{1}{2}\,(\mathrm{dist}_U(\cdot))^2$ (which is convex and continuous) and $\Phi_2 = (\Phi + I_K) \in \Gamma_0(\mathbb{R}^n)$, we get

$$\Psi^* = \left[\frac{1}{2}\mathrm{dist}_{\ker(M)}(\cdot)^2\right]^* \,\square\, [\Phi + I_K]^*.$$

Hence

$$\mathrm{Dom}\ \Psi^* = \mathrm{Dom}\left[\frac{1}{2}\mathrm{dist}_{\ker(M)}(\cdot)^2\right]^* + \mathrm{Dom}\ [\Phi + I_K]^*.$$

On the other hand, we have:

$$\frac{1}{2}\mathrm{dist}_{\ker(M)}(\cdot)^2 = \frac{1}{2}\|\cdot\|^2 \,\square\, I_{\ker(M)}.$$

Using (8) again, we obtain

$$\left[\frac{1}{2}\mathrm{dist}_{\ker(M)}(\cdot)^2\right]^* = \frac{1}{2}\|\cdot\|^2 + I_{\ker(M)^\perp}.$$

Hence

$$\mathrm{Dom}\left[\mathrm{dist}_{\ker(M)}(\cdot)^2\right]^* = \ker(M)^\perp.$$

Consequently,

$$\mathrm{Dom}\ \Psi^* = R(M) + \mathrm{Dom}\ [\Phi + I_K]^*. \tag{11}$$

□

Proposition 2 *Suppose that assumptions* $(\mathcal{H})$ *hold. Then a necessary condition for the existence of a solution of* $\mathrm{VI}(M, q, \Phi, K)$ *is that*

$$\langle q, w\rangle + \Phi_\infty(w) \geq 0, \quad \forall w \in \ker(M) \cap K_\infty. \tag{12}$$

Proof Using Remark 1 and Lemma 1, we have

$$\mathcal{R}(M, \Phi, K) \subset \overline{\text{Dom}\,(\Psi^*)}.$$

Using Part (i) of Proposition 1, we have

$$\overline{\text{Dom}\,(\Psi^*)} = \{g \in \mathbb{R}^n \ : \ \langle g, w \rangle \leq \Psi_\infty(w), \quad \forall w \in \mathbb{R}^n\}.$$

It can also easily be checked that the recession function Ψ_∞ associated to Ψ in (9) is given by

$$\Psi_\infty(w) = I_{\ker(M)}(w) + \Phi_\infty(w) + I_{K_\infty}(w). \tag{13}$$

Consequently if $\text{Sol}(M, q, \Phi, K) \neq \emptyset$ then $-q \in \mathcal{R}(M, \Phi, K)$ and thus:

$$\langle q, w \rangle + \Phi_\infty(w) \geq 0, \quad \forall w \in \ker(M) \cap K_\infty$$

which competes the proof of Proposition 2. □

Proposition 3 *Let C be a non-empty closed convex subet of $\mathbb{R}^n$. We have*

$$\text{Int}\,\mathcal{B}(C) \neq \emptyset \Longleftrightarrow C_\infty \cap (-C_\infty) = \{0\}, \tag{14}$$

where $\mathcal{B}(C)$ denotes the barrier cone of C defined in (3).

Proof Since $\mathcal{B}(C) = \text{Dom}\,(I_C)^*$, we have

$$\text{Int}\,\mathcal{B}(C) \neq \emptyset \Longleftrightarrow \text{Int Dom}\,(I_C)^* \neq \emptyset.$$

Using Part (ii) of Proposition 1 for $\Psi = I_C$, we have

$$\text{Int Dom}\,(I_C)^* = \{g \in \mathbb{R}^n \ : \ \langle g, w \rangle < I_{C_\infty}(w), \quad \forall w \in \mathbb{R}^n, \ w \neq 0\}.$$

Hence,

$$\text{Int}\,\mathcal{B}(C) = \{g \in \mathbb{R}^n \ : \ \langle g, w \rangle < 0, \ \forall w \in C_\infty, \ w \neq 0\}.$$

Therefore,

$$\text{Int}\,\mathcal{B}(C) = \text{Int}\,(C_\infty)^\circ.$$

Consequently, if $\text{Int}\,\mathcal{B}(C) \neq \emptyset$ then there exists $g \in \mathbb{R}^n$ such that:

$$\langle g, w \rangle < 0, \quad \forall w \in C_\infty, \quad w \neq 0,$$

which implies that necessarily $C_\infty \cap -C_\infty = \{0\}$.

Conversely, suppose that $C_\infty \cap -C_\infty = \{0\}$. Arguing by contradiction, let us suppose that $\text{Int}\,(C_\infty)^\circ = \emptyset$. Since $(C_\infty)^\circ$ is cone, then there exists a linear subspace E of $\mathbb{R}^n$ with $0 \leq \dim_{\mathbb{R}}(E) \leq n-1$ such that: $(C_\infty)^\circ \subset E$ (see e.g. [14], p. 33). Therefore, $E^\perp \subset (C_\infty)^{\circ\circ} = C_\infty$, where $E^\perp$ is the orthogonal of E. This contradicts the assumption $C_\infty \cap -C_\infty = \{0\}$ (since $\dim_{\mathbb{R}} E^\perp \geq 1$ and $E^\perp \subset C_\infty$). Hence, $\text{Int}\,(C_\infty)^\circ \neq \emptyset$. Consequently, $\text{Int}\,\mathcal{B}(C) \neq \emptyset$, which completes the proof of Proposition 3. □

Remark 2 A subset C satisfying the condition $\text{Int}\,\mathcal{B}(C) \neq \emptyset$ is called *well-positioned* (see Proposition 2.1 [2]). An other characterization of this class of convex subsets in infinite dimensional spaces is also given in Lemma 2.4 [2] and Proposition 2.1 [3]. Note that in finite dimensional spaces every subset which contains no line is *well-positioned*.

Definition 1 The set defined by

$$\left\{q \in \mathbb{R}^n \ : \ \langle q, w\rangle + \Phi_\infty(w) > 0, \quad \forall w \in \ker(M) \cap K_\infty \setminus \{0\}\right\},$$

is called the compatibility set of the variational inequality $\mathrm{VI}(M, q, \Phi, K)$.

The following proposition shows that the topological interior of the resolvent set coincides with the compatibility set of the variational inequality $\mathrm{VI}(M, q, \Phi, K)$.

Proposition 4 *Suppose that assumptions* $(\mathcal{H})$ *hold. We have*

$$\mathrm{Int}\,\mathcal{R}(M, \Phi, K) = \left\{q \in \mathbb{R}^n \ : \ \langle q, w\rangle + \Phi_\infty(w) > 0, \quad \forall w \in \ker(M) \cap K_\infty \setminus \{0\}\right\}.$$

Proof Using Remark 1 and Lemma 1, we have $\mathrm{Int}\,\mathcal{R}(M, \Phi, K) \subset \mathrm{Int\ Dom}\ \Psi^*$.
By part (ii) of Proposition 1, we have

$$\mathrm{Int\,Dom}\,(\Psi^*) = \left\{g \in \mathbb{R}^n \ : \ \langle g, w\rangle < \Psi_\infty(w), \quad \forall w \in \mathbb{R}^n \setminus \{0\}\right\}.$$

Using (13), we get

$$\mathrm{Int}\,\mathcal{R}(M, \Phi, K) \subset \left\{q \in \mathbb{R}^n \ : \ \langle q, w\rangle + \Phi_\infty(w) > 0, \quad \forall w \in \ker(M) \cap K_\infty \setminus \{0\}\right\}.$$

Let us prove now the converse, i.e., let $q \in \mathbb{R}^n$ such that

$$\langle q, w\rangle + \Phi_\infty(w) > 0, \quad \forall w \in \ker(M) \cap K_\infty \setminus \{0\}. \tag{15}$$

We prove that $-q \in \mathcal{R}(M, \Phi, K)$. Let $(\varepsilon_n)_n$ be a sequence of non-negative real numbers such that $\varepsilon_n \to 0^+$ as $n \to +\infty$. Let $u_n \in K$ be the unique solution of problem $\mathrm{VI}(M + \varepsilon_n I, \Phi, q, K)$. We claim that the sequence (u_n) is bounded. Arguing by contradiction, suppose that there exists a subsequence, still denoted (u_n) such that $\|u_n\| \to +\infty$ as $n \to +\infty$. We set $w_n = \dfrac{u_n}{\|u_n\|}$ and along a subsequence, we may suppose that $w_n \to w \neq 0$. It is clear that $w \in K_\infty$. We have

$$\langle Mu_n + q, v - u_n\rangle + \varepsilon_n\langle u_n, v - u_n\rangle + \Phi(v) - \Phi(u_n) \geq 0, \quad \forall v \in K. \tag{16}$$

Setting $v = 0$ in (16), we get

$$\langle Mu_n + q, u_n\rangle + \varepsilon_n\|u_n\|^2 + \Phi(u_n) - \Phi(0) \leq 0.$$

Dividing by $\|u_n\|^2$ and passing to the limit as $n \to +\infty$, we get $\langle Mw, w\rangle \leq 0$. Since M is symmetric and positive semidefinite, we have $w \in \ker(M)$ and consequently $w \in \ker(M) \cap K_\infty \setminus \{0\}$.
On the other hand dividing (16) by $t_n = \|u_n\|$ and using the fact that $\langle Mw, w\rangle \geq 0$, we obtain

$$\langle q, w_n\rangle + \frac{\Phi(t_n w_n)}{t_n} - \frac{\Phi(0)}{t_n} \leq 0.$$

Passing to the limit, we get

$$\langle q, w\rangle + \Phi_\infty(w) \leq 0,$$

which is a contradiction to the condition (15). Thus the sequence (u_n) is bounded and there exists a subsequence, again denoted (u_n) such that $u_n \to u$ as $n \to +\infty$. Passing to the limit as $n \to +\infty$ in (16), we show that u is a solution of $\mathrm{VI}(M, q, \Phi, K)$. Hence $-q \in \mathcal{R}(M, \Phi, K)$. Therefore,

$$\mathrm{Int\,Dom}\,(\Psi^*) \subset \mathcal{R}(M, \Phi, K),$$

which implies that

$$\mathrm{Int}\,\mathrm{Dom}\,(\Psi^*) \subset \mathrm{Int}\,\mathcal{R}(M, \Phi, K).$$

Consequently,

$$\mathrm{Int}\,\mathcal{R}(M, \Phi, K) = \mathrm{Int}\,\mathrm{Dom}\,(\Psi^*) = \big\{q \in \mathbb{R}^n \;:\; \langle q, w\rangle + \Phi_\infty(w) > 0, \quad \forall w \in \ker(M) \cap K_\infty \setminus \{0\}\big\},$$

which completes the proof of Proposition 4. □

The following result characterizes the non-emptyness of the topological interior of the resolvent set $\mathcal{R}(M, \Phi, K)$ associated to the variational inequality $\mathrm{VI}(M, q, \Phi, K)$.

Proposition 5 *Suppose that assumptions* $(\mathcal{H})$ *hold. We have*

$$\mathrm{Int}\,\mathcal{R}(M, \Phi, K) \neq \emptyset \Longleftrightarrow -\left(\ker(M) \cap K_\infty \cap \ker(\Phi_\infty)\right) \bigcap \left(\ker(M) \cap K_\infty \cap \ker(\Phi_\infty)\right) = \{0\},$$

i.e., the cone $(\ker(M) \cap K_\infty \cap \ker(\Phi_\infty))$ *is pointed.*

Proof We have

$$\mathrm{Int}\,\mathcal{R}(M, \Phi, K) \neq \emptyset \Longleftrightarrow \mathrm{Int}\,\mathrm{Dom}\,(\Psi^*) \neq \emptyset. \tag{17}$$

On the other hand, we have

$$\left(\mathrm{Dom}\,(\Psi^*)\right) \times \{-1\} = \mathcal{B}(\mathrm{epi}\,\Psi) \cap \left(\mathbb{R}^n \times \{-1\}\right), \tag{18}$$

where $\mathcal{B}(\mathrm{epi}\,\Psi)$ is the barrier cone to $\mathrm{epi}\,(\Psi)$.
Indeed, let $p \in \mathrm{Dom}\ \ \Psi^*$ be given. Then there exists $M_p \in \mathbb{R}$ such that

$$\langle p, x\rangle - \Psi(x) \leq M_p, \quad \forall x \in \mathbb{R}^n,$$

which is equivalent to

$$\Big\langle (p, -1), (x, \Psi(x)) \Big\rangle_{\mathbb{R}^n \times \mathbb{R}} \leq M_p, \quad \forall x \in \mathbb{R}^n.$$

Therefore,

$$\Big\langle (p, -1), (x, \lambda) \Big\rangle_{\mathbb{R}^n \times \mathbb{R}} \leq M_p, \quad \forall (x, \lambda) \in \ \mathrm{epi}\,(\Psi).$$

Hence,

$$(p, -1) \in \mathcal{B}(\mathrm{epi}\,\Psi).$$

Now let $(p, -1) \in \mathcal{B}(\mathrm{epi}\,\Psi) \cap (\mathbb{R}^n \times \{-1\})$. Then there existe $M_p \in \mathbb{R}^n$ such that

$$\Big\langle (p, -1), (x, \lambda) \Big\rangle_{\mathbb{R}^n \times \mathbb{R}} \leq M_p, \quad \forall (x, \lambda) \in \ \mathrm{epi}\,(\Psi).$$

In particular for $(x, \lambda) = (x, \Psi(x))$, we have

$$\langle p, x\rangle - \Psi(x) \leq M_p, \quad \forall x \in \mathbb{R}^n.$$

Consequently,

$$(p, -1) \in \left(\mathrm{Dom}\,(\Psi^*)\right) \times \{-1\}.$$

Using (17) and (18), we have

$$\operatorname{Int} \mathcal{R}(M, \Phi, K) \neq \emptyset \Longleftrightarrow \operatorname{Int} \mathcal{B}(\operatorname{epi} \Psi) \neq \emptyset. \tag{19}$$

Using Proposition 3 for $C = \operatorname{epi} \Psi$, we have

$$\operatorname{Int} \mathcal{B}(\operatorname{epi} \Psi) \neq \emptyset \Longleftrightarrow ((\operatorname{epi} \Psi)_\infty) \cap (-(\operatorname{epi} \Psi)_\infty) = \{0\}. \tag{20}$$

Since, $(\operatorname{epi} \Psi)_\infty = \operatorname{epi} \Psi_\infty$ we get

$$\operatorname{Int} \mathcal{R}(M, \Phi, K) \neq \emptyset \Longleftrightarrow ((\operatorname{epi} \Psi_\infty)) \cap (-(\operatorname{epi} \Psi_\infty)) = \{0\}. \tag{21}$$

Since Φ is bounded from below, then Ψ is also bounded from below. Hence, $\Psi_\infty \geq 0$ and it is then easy to check that:

$$((\operatorname{epi} \Psi_\infty)) \cap (-(\operatorname{epi} \Psi_\infty)) = ((\ker \Psi_\infty) \times \{0\}) \cap (-(\ker \Psi_\infty) \times \{0\})\,.$$

Therefore,

$$\operatorname{Int} \mathcal{R}(M, \Phi, K) \neq \emptyset \Longleftrightarrow ((\ker \Psi_\infty) \times \{0\}) \cap (-(\ker \Psi_\infty) \times \{0\}) = \{0\}. \tag{22}$$

Consequently,

$$\operatorname{Int} \mathcal{R}(M, \Phi, K) \neq \emptyset \Longleftrightarrow ((\ker \Psi_\infty)) \cap (-(\ker \Psi_\infty)) = \{0\}. \tag{23}$$

Using (13), it is easy to see that

$$\begin{aligned}\operatorname{Int} \mathcal{R}(M, \Phi, K) \neq \emptyset \Longleftrightarrow &-(\ker(M) \cap K_\infty \cap \ker(\Phi_\infty)) \bigcap (\ker(M) \cap K_\infty \cap \ker(\Phi_\infty)) \\ &= \{0\},\end{aligned} \tag{24}$$

which completes the proof of Proposition 5. □

4 A stability result

Let us now study the stability of the variational inequality $\mathrm{VI}(M, q, \Phi, K)$ in the following sense.

Definition 2 One says that the variational inequality $\mathrm{VI}(M, q, \Phi, K)$ is stable provided that there exists $\varepsilon > 0$ such that for any symmetric and positive semidefinite matrix M_ε, any vector $q_\varepsilon \in q + \varepsilon \mathbb{B}_n$ (here $\mathbb{B}_n$ denotes the open unit ball in $\mathbb{R}^n$), any proper lower semicontinuous bounded from below convex function Φ_ε, and any non-empty closed convex set K_ε satisfying the following conditions

$$0 \in \operatorname{Dom} \Phi_\varepsilon \cap K_\varepsilon \text{ and } \ker(M) \cap \ker(\Phi_\infty) \cap K_\infty = \ker(M_\varepsilon) \cap \ker((\Phi_\varepsilon)_\infty) \cap (K_\varepsilon)_\infty, \tag{25}$$

the perturbed problem $VI(M_\varepsilon, q_\varepsilon, \Phi_\varepsilon, K_\varepsilon)$ has at least one solution.

Before starting our study, let us first give some simple examples to motivate the notion of stability of $\mathrm{VI}(M, q, \Phi, K)$ with respect to small perturbations.

Example 1 Set

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad K_1 = \{(x, y) \in \mathbb{R}^2 \ : \ y \geq 0\}, \quad \Phi \equiv 0.$$

In this case the resolvent set is given by

$$\mathcal{R}(M, 0, K_1) = \{(q_1, q_2) \in \mathbb{R}^2 \ : \ q_1 = 0 \text{ and } q_2 \leq 0\}.$$

We note that in this case Int $\mathcal{R}(M, 0, K_1) = \emptyset$. Hence problem VI$(M, q, 0, K_1)$ is not stable with respect to small perturbations of the right hand term q. More precisely, if we replace q by q_ε such that $\|q - q_\varepsilon\| \leq \varepsilon$ for a given $\varepsilon > 0$, the solution set Sol$(M, q_\varepsilon, 0, K_1)$ of the perturbed problem may be empty.
We note that in this case, we have

$$\ker(M) \cap K_{1\infty} \cap \ker(\Phi_\infty) = \mathbb{R} \times \mathbb{R}^+,$$

which is not a pointed cone.

Example 2 Let M and Φ as in Example 1 Consider now, the convex and closed subset K_2 given by

$$K_2 = \{(x, y) \in \mathbb{R}^2 \ : \ y \geq x^2\}.$$

In this case the resolvent set is given by

$$\mathcal{R}(M, 0, K_2) = \{(q_1, q_2) \in \mathbb{R}^2 \ : \ q_2 < 0\} \cup \{(0, 0)\}.$$

Here Int $\mathcal{R}(M, 0, K_2) \neq \emptyset$ and problem $VI(M, q, 0, K_2)$ is stable with respect to small perturbations of the right hand term q.
We note that in this case, we have

$$\ker(M) \cap K_{2\infty} \cap \ker(\Phi_\infty) = \{0\} \times \mathbb{R}^+,$$

which is a pointed cone.

We have the following existence and stability result related to the linear variational inequality VI(M, q, Φ, K).

Theorem 1 *Assume that assumptions* $(\mathcal{H})$ *are satisfied. Then the variational inequality* VI(M, q, Φ, K) *is stable in the sense of Definition 2 if and only if*

$$\langle q, w\rangle + \Phi_\infty(w) > 0, \quad \forall w \in \ker(M) \cap K_\infty, \quad w \neq 0.$$

Proof We know from Proposition 4 that if

$$\langle q, w\rangle + \Phi_\infty(w) > 0, \quad \forall w \in \ker(M) \cap K_\infty \setminus \{0\}$$

then

$$-q \in \text{Int}\, \mathcal{R}(M, \Phi, K) \neq \emptyset.$$

Therefore, there exists $\varepsilon > 0$ such that

$$-q + \varepsilon \mathbb{B}_n \subset \text{Int}\, \mathcal{R}(M, \Phi, K). \tag{26}$$

Let now M_ε be a symmetric and positive semidefinite matrix, $q_\varepsilon \in q + \varepsilon \mathbb{B}_n$, Φ_ε be a proper lower semicontinuous bounded from below convex function and K_ε be a non-empty closed convex set satisfying

$$0 \in \text{Dom } \Phi_\varepsilon \cap K_\varepsilon$$

and

$$\ker(M) \cap \ker(\Phi_\infty) \cap K_\infty = \ker(M_\varepsilon) \cap \ker((\Phi_\varepsilon)_\infty) \cap (K_\varepsilon)_\infty.$$

Using Proposition 4, we have

$$\operatorname{Int} \mathcal{R}(M, \Phi, K) = \operatorname{Int} \mathcal{R}(M_\varepsilon, \Phi_\varepsilon, K_\varepsilon).$$

Therefore,

$$-q_\varepsilon \in -q + \varepsilon\mathbb{B} \subset \operatorname{Int} \mathcal{R}(M_\varepsilon, \Phi_\varepsilon, K_\varepsilon).$$

Consequently, $\mathrm{Sol}(M_\varepsilon, q_\varepsilon, \Phi_\varepsilon, K_\varepsilon) \neq \emptyset$. This ensures that the variational inequality $\mathrm{VI}(M, q, \Phi, K)$ is stable in the sense of Definition 2.

Suppose now that the variational inequality $\mathrm{VI}(M, q, \Phi, K)$ is stable in the sense of Definition 2. Then there exists $\varepsilon > 0$ such that (taking $M_\varepsilon = M$, $\Phi_\varepsilon = \Phi$ and $K_\varepsilon = K$) for every $q_\varepsilon \in q + \varepsilon\mathbb{B}_n$, $\mathrm{Sol}(M, q_\varepsilon, \Phi, K) \neq \emptyset$. Therefore,

$$-q \in \operatorname{Int} \mathcal{R}(M, \Phi, K).$$

Then using Proposition 4, we obtain

$$\langle q, w\rangle + \Phi_\infty(w) > 0, \quad \forall w \in \ker(M) \cap K_\infty, \quad w \neq 0, \tag{27}$$

which completes the proof of Theorem 1. □

Remark 3 It is clear that any symmetric and positive semidefinite matrix M_ε such that $\ker(M) = \ker(M_\varepsilon)$ and any function $\Phi_\varepsilon \in \Gamma_0(\mathbb{R}^n)$ bounded from below such that $\Phi_\infty = (\Phi_\varepsilon)_\infty$ and any non-empty closed convex set K_ε such that $K_\infty = (K_\varepsilon)_\infty$ satisfy condition (25). This the case for example if we take $M = M_\varepsilon$, $\Phi - \varepsilon \leq \Phi_\varepsilon \leq \Phi + \varepsilon$, $K \subset K_\varepsilon + \varepsilon\mathbb{B}_n$ and $K_\varepsilon \subset K + \varepsilon\mathbb{B}_n$.

We have the following consequence of Theorem 1.

Corollary 1 *Assume that assumptions* ($\mathcal{H}$) *are satisfied. If the following compatibility condition*

$$\langle q, w\rangle + \Phi_\infty(w) > 0, \quad \forall w \in \ker(M) \cap K_\infty, \quad w \neq 0, \tag{28}$$

is satisfied, then $\mathrm{VI}(M, q, \Phi, K)$ *has at least one solution.*

Proof Take $M_\varepsilon = M$, $\Phi_\varepsilon = \Phi$, $q_\varepsilon = q$ and $K_\varepsilon = K$ in Theorem 1. □

5 Some applications

We give in this section some applications of Theorem 1 and Corollary 1.

Example 3 (Clipping circuit 1/Ideal diode) Let us consider the circuit of Fig. 1 involving a load resistance $R > 0$, an input-signal source u and corresponding instantaneous current i, an ideal diode as a shunt element and a supply voltage E.

Figure 2 illustrates the ampere-volt characteristic of an ideal diode.

This is a model in which the diode is a simple switch. If $V < 0$ then $i = 0$ and the diode is blocking. If $i > 0$ then $V = 0$ and the diode is conducting. We first see that the ideal diode is described by the complementarity relation

$$V \leq 0, \quad i \geq 0, \quad Vi = 0$$

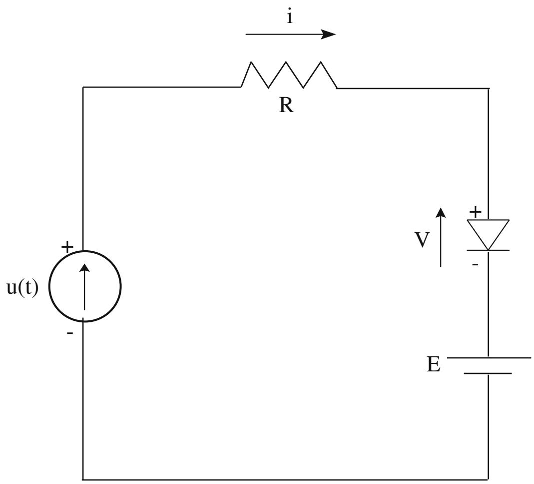
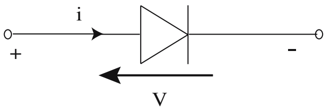


Fig. 1 Clipping circuit 1: diode as shunt element



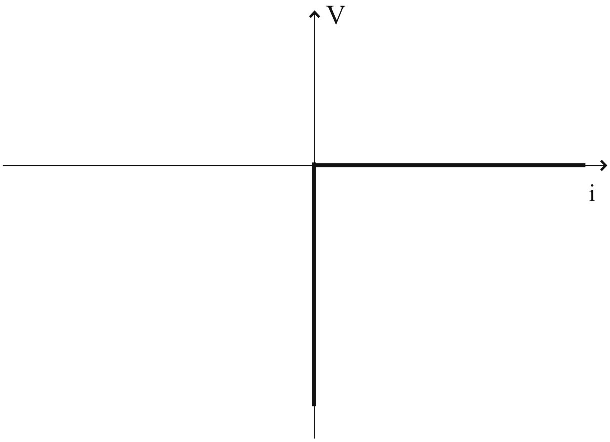


Fig. 2 Ideal diode model

that is also

$$\min\{-V, i\} = 0.$$

The electrical superpotential of the ideal diode is

$$\varphi_D(x) = I_{\mathbb{R}_+}(x), \quad (x \in \mathbb{R})$$

Then

$$\varphi_D^*(z) = I_{\mathbb{R}_-}(z), \quad (z \in \mathbb{R})$$

and the recession function of the electrical superpotential is:

$$(\varphi_D)_\infty(x) = \varphi_D(x), \ (x \in \mathbb{R}).$$

We have also

$$\partial\varphi_D(x) := \begin{cases} \mathbb{R}_- & \text{if } x = 0 \\ 0 \text{ if } x > 0, \\ \emptyset \text{ if } x < 0 \end{cases} \qquad (x \in \mathbb{R})$$

and

$$\partial\varphi_D^*(z) := \begin{cases} \mathbb{R}_+ & \text{if } z = 0 \\ 0 \text{ if } z < 0, \\ \emptyset \text{ if } z > 0 \end{cases} \qquad (z \in \mathbb{R}).$$

The complementarity relation can be written as

$$V \in \partial\varphi_D(i) \Longleftrightarrow i \in \partial\varphi_D^*(V) \Longleftrightarrow \varphi_D(i) + \varphi_D^*(V) = iV.$$

Kirchoff's voltage law gives

$$u = U_R + V_D + E$$

where $U_R = Ri$ denotes the difference of potential across the resistor and $V_D \in \partial I_{\mathbb{R}_+}(i)$ is the difference of potential across diode. Thus

$$E + Ri - u \in -\partial I_{\mathbb{R}_+}(i) \tag{29}$$

which is equivalent to **VI**(**R**, **E** − **u**, **0**, $\mathbb{R}_+$), i.e.,

$$i \in \mathbb{R}_+ : (Ri + E - u)(v - i) \geq 0, \quad \forall v \in \mathbb{R}_+. \tag{30}$$

Here $R > 0$ and for each $E, u \in \mathbb{R}$, we may apply Theorem 1 to assert that (30) is stable in the sense of Definition 2.
Moreover:

$$(30) \Longleftrightarrow \frac{E}{R} + i - \frac{u}{R} \in -\partial I_{\mathbb{R}_+}(i) \Longleftrightarrow -\frac{E}{R} + \frac{u}{R} \in i + \partial I_{\mathbb{R}_+}(i)$$
$$\Longleftrightarrow i = (id_{\mathbb{R}} + \partial I_{\mathbb{R}_+})^{-1}(\frac{u - E}{R}) = \frac{1}{R}\max\{0, u - E\}.$$

If $u \leq E$ then the diode is blocking while if $u > E$ then the diode is conducting.
Let us now consider a driven time depending input $t \mapsto u(t)$ and define the output-signal $t \mapsto V_o(t)$ as

$$V_o(t) = E + V(t).$$

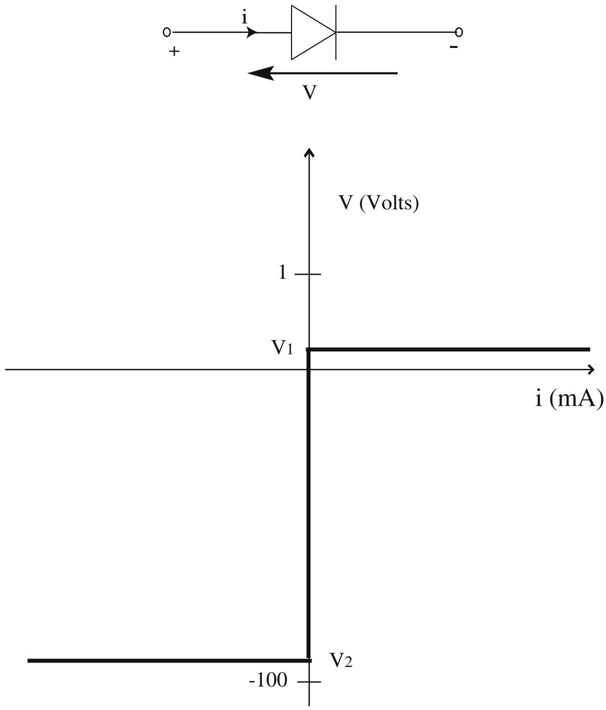
The time depending current $t \mapsto i(t)$ is given by

$$i(t) = \frac{1}{R}\max\{0, u(t) - E\} \tag{31}$$

and thus

$$V_o(t) = V(t) + E = u(t) - Ri(t) = u(t) + \min\{0, E - u(t)\} = \min\{u(t), E\}. \tag{32}$$

Fig. 3 Practical diode model



This shows that the circuit in Fig. 1 can be used to transmit the part of a given input-signal u which lies below some given reference level E.

Example 4 (*Clipping circuit 1/concrete diode*) Let us now consider the circuit in Fig. 1 with a concrete diode.

Figure 3 illustrates the ampere–volt characteristic of a practical diode model.

There is a voltage point, called the knee voltage V_1, at which the diode begins to conduct and a maximum reverse voltage, called the peak reverse voltage V_2, that will not force the diode to conduct. When this voltage is exceeded, the depletion may breakdown and allow the diode to conduct in the reverse direction. Note that usually $\mid V_1 \mid \ll \mid V_2 \mid$ and the model is locally ideal.

For general purpose diodes used in low frequency/speed applications, $\mid V_1 \mid \simeq 0.7-2.5\,\text{V}$ and $\mid V_2 \mid \simeq 5\,\text{kV}$; for high voltage rectifier diodes, $\mid V_1 \mid \simeq 10\,\text{V}$ and $\mid V_2 \mid \simeq 30\,\text{kV}$; for fast diodes used in switched mode power supply and inverter circuits, $\mid V_1 \mid \simeq 0.7 - 1.5\,\text{V}$ and $\mid V_2 \mid \simeq 3\,\text{kV}$ and for Schottky diodes used in high frequency applications, $\mid V_1 \mid \simeq 0.2 - 0.9\,\text{V}$ and $\mid V_2 \mid \simeq 100\,\text{V}$.
The electrical superpotential of the practical diode is

$$\varphi_{PD}(x) = \begin{cases} V_1 x & \text{if } x \geq 0 \\ V_2 x & \text{if } x < 0 \end{cases}, \quad (x \in \mathbb{R}).$$

Then

$$\varphi^*_{PD}(z) = I_{[V_2, V_1]}(z), \quad (z \in \mathbb{R})$$

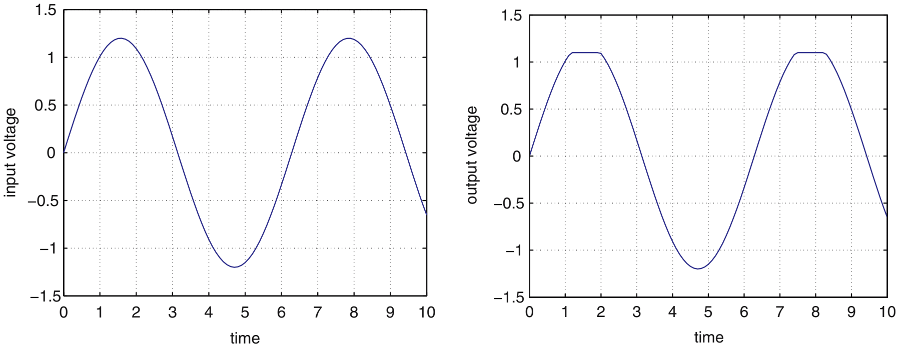


Fig. 4 Clipping circuit 1: general diode as shunt element using, $V_1 = 0.1$, $V_2 = -90$, $E = 1$

and the recession function of the electrical superpotential is given by:

$$(\varphi_{PD})_\infty(x) = \varphi_{PD}(x), \quad (x \in \mathbb{R}).$$

We see that

$$\partial\varphi_{PD}(x) = \begin{cases} V_2 \text{ if } x < 0 \\ [V_2, V_1] \text{ if } x = 0 \\ V_1 \text{ if } x > 0 \end{cases}, \quad (x \in \mathbb{R})$$

recovers the ampere–volt characteristic (i, V) while

$$\partial\varphi^*_{PD}(z) = \begin{cases} \mathbb{R}_- \text{ if } z = V_2 \\ 0 \text{ if } z \in]V_2, V_1[\\ \mathbb{R}_+ \text{ if } z = V_1 \\ \emptyset \text{ if } z \in \mathbb{R}\setminus[V_2, V_1] \end{cases}, \quad (z \in \mathbb{R}).$$

recovers the volt–ampere characteristic (V, i). The ampere–volt characteristic of the practical diode (Fig. 4) can thus be written as

$$V \in \partial\varphi_{PD}(i) \Longleftrightarrow i \in \partial\varphi^*_{PD}(V) \Longleftrightarrow \varphi_{PD}(i) + \varphi^*_{PD}(V) = iV.$$

We may follow the same steps as in the previous example to see that Kirchoff's law reduces to $\mathbf{VI}(\mathbf{R}, \mathbf{E} - \mathbf{u}, \varphi_{\mathbf{PD}}, \mathbb{R})$, i.e.,

$$i \in K := \mathbb{R} : (Ri + E - u)(v - i) + \varphi_{PD}(v) - \varphi_{PD}(i) \geq 0, \quad \forall v \in \mathbb{R}.$$

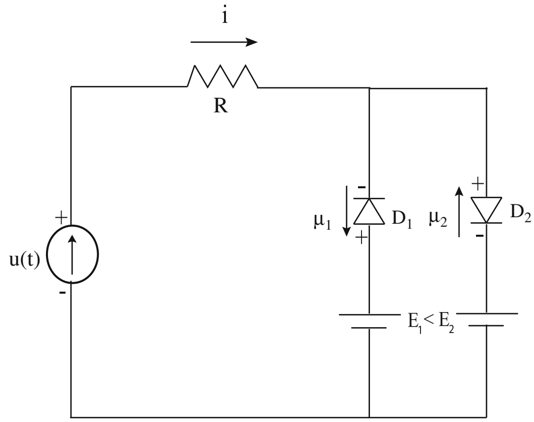
Here $R > 0$ and for each $E, u \in \mathbb{R}$, we may apply Theorem 1 to assert that (4) is stable in the sense of Definition 2. Moreover:

$$i(t) = (id_{\mathbb{R}} + \partial\varphi_{PD})^{-1}\left(\frac{u(t) - E}{R}\right) = \text{argmin}_{x \in \mathbb{R}}\left\{\frac{1}{2}\left|x - \left(\frac{u(t) - E}{R}\right)\right|^2 + \varphi_{PD}(x)\right\}. \tag{33}$$

and

$$V_o(t) = u(t) - Ri(t). \tag{34}$$

Fig. 5 Double-diode clipper



Example 5 Let us consider the double-diode clipper circuit in Fig. 5 involving a load resistance $R > 0$, two ideal diodes, an input-signal source and two supply voltages E_1 and E_2. It is assumed that $E_1 < E_2$. We denote by i the current through the resistor R and we set $i = i_1 + i_2$ where $-i_1$ denotes the current through diode D_1 and i_2 is the current through diode D_2. We denote by μ_1 the difference of potential across diode D_1 and by μ_2 the difference of potential across diode D_2.

Using Kirchoff's voltage laws, we get the system:

$$\begin{cases} E_1 + R(i_1 + i_2) - u = +\mu_1 \\ E_2 + R(i_1 + i_2) - u = -\mu_2 \end{cases} \tag{35}$$

The ideal diodes D_1 and D_2 are simple switch. If $\mu_1 < 0$ (resp. $\mu_2 < 0$) then $-i_1 = 0$ (resp. $i_2 = 0$) and the diode is blocking while if $-i_1 > 0$ (resp. $i_2 > 0$) then $\mu_1 = 0$ (resp. $\mu_2 = 0$) and the diode is conducting. Thus

$$\mu_1 \leq 0, \quad -i_1 \geq 0, \quad \mu_1 i_1 = 0$$

and

$$\mu_2 \leq 0, \quad -i_2 \geq 0, \quad \mu_2 i_2 = 0.$$

These last complementarity relations can also be written as:

$$\mu_1 \in \partial I_{\mathbb{R}_+}(-i_1) = -I_{\mathbb{R}_-}(i_1)$$

and

$$\mu_2 \in \partial I_{\mathbb{R}_+}(i_2)$$

Setting

$$K = \mathbb{R}_- \times \mathbb{R}_+, \quad M = \begin{pmatrix} R & R \\ R & R \end{pmatrix}, \quad q = \begin{pmatrix} E_1 - u \\ E_2 - u \end{pmatrix}, \quad I = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}, \tag{36}$$

we see that the system in (35) is equivalent to the variational inequality $\mathbf{VI}(\mathbf{M}, \mathbf{q}, \mathbf{0}, \mathbf{K})$, i.e.,

$$I \in K : \langle MI + q, v - I \rangle \geq 0, \quad \forall v \in K. \tag{37}$$

Here the matrix M is positive semidefinite and symmetric,

$$K_\infty \equiv K,\ \ker\{M\} = \{v \in \mathbb{R}^2 : v_2 = -v_1\}$$

and

$$K_\infty \cap \ker\{M\} = \{(-\alpha, \alpha); \alpha \geq 0\}.$$

Moreover, for all $v \in K_\infty \cap \ker\{M\}$, $v \neq (0, 0)$ there exists $\alpha > 0$ such that $v = (-\alpha, \alpha)$ and

$$\langle q, v\rangle = (E_1 - u)v_1 + (E_2 - u)v_2 = \alpha(E_2 - E_1) > 0. \tag{38}$$

We may thus apply Theorem 1 to assert that a double-diode clipper involving ideal diodes is stable with respect to data perturbations in the sense of Definition 2.
Moreover, using the relations in (35) we see that:

$$i_1^* + i_2^* = \min\left\{i_2^*, \frac{u - E_1}{R}\right\} = \max\left\{i_1^*, \frac{u - E_2}{R}\right\}$$

from which we deduce, after elementary calculations, that:

$$i^* = \begin{cases} \frac{u-E_1}{R} \text{ if } u < E_1 \\ 0 \text{ if } E_1 \leq u \leq E_2 \\ \frac{u-E_2}{R} \text{ if } u > E_2 \end{cases}.$$

So, for a driven time depending input $t \mapsto u(t)$ the time depending current $t \mapsto i^*(t)$ through the resistor R is given by

$$i^*(t) = \begin{cases} \frac{u(t)-E_1}{R} \text{ if } u(t) < E_1 \\ 0 \text{ if } E_1 \leq u(t) \leq E_2 \\ \frac{u(t)-E_2}{R} \text{ if } u(t) > E_2 \end{cases} \tag{39}$$

and the output-signal $t \mapsto V_o(t)$ defined by

$$V_o(t) = V_2(t) + E_2 = u(t) - Ri^*(t)$$

is then given by the expression:

$$V_o(t) = \begin{cases} E_1 \text{ if } u(t) < E_1 \\ u(t) \text{ if } E_1 \leq u(t) \leq E_2 \\ E_2 \text{ if } u(t) > E_2 \end{cases}. \tag{40}$$

This shows that the circuit can be used to transmit the part of a given input-signal u that lies above some level E_1 and below some level E_2 (Fig. 6).

Example 6 Let us again consider the circuit in Fig. 5 and suppose that the electrical super-potential of each diodes D_1 and D_2 is given by (practical diode model):

$$\varphi_{PD}(x) = \begin{cases} V_1 x \text{ if } x \geq 0 \\ V_2 x \text{ if } x < 0 \end{cases}, \quad (x \in \mathbb{R})$$

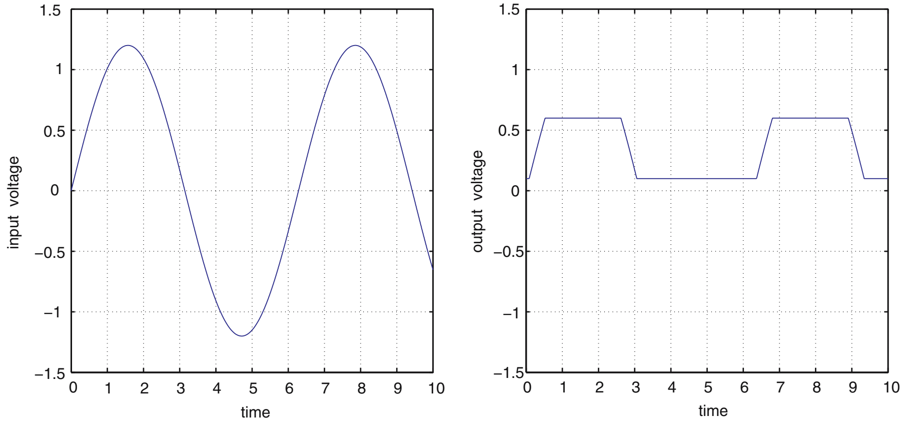


Fig. 6 Double-diode clipper: ideal diode, $E_1 = 0.1$, $E_2 = 0.6$

where $V_2 < 0 < V_1$. We suppose also that

$$| V_2 | > \frac{E_2 - E_1}{2}. \tag{41}$$

We set

$$\bar{\varphi}_{PD}(x) = \varphi_{PD}(-x), \quad \forall\, x \in \mathbb{R}$$

and

$$\Phi(x) = \bar{\varphi}_{PD}(x_1) + \varphi_{PD}(x_2), \quad \forall\, (x_1, x_2) \in \mathbb{R}^2. \tag{42}$$

Kirchoff's laws yield the system

$$\begin{cases} E_1 + R(i_1 + i_2) - u = +\mu_1 \in -\partial\bar{\varphi}_{PD}(i_1) \\ \\ E_2 + R(i_1 + i_2) - u = -\mu_2 \in -\partial\varphi_{PD}(i_2) \end{cases} \tag{43}$$

which is equivalent to the variational inequality $\mathbf{VI}(\mathbf{M}, \mathbf{q}, \Phi, \mathbb{R}^2)$, i.e.,

$$\Upsilon \in \mathbb{R}^2 : \langle M\Upsilon + q, v - \Upsilon\rangle + \Phi(v) - \Phi(\Upsilon) \geq 0, \quad \forall v \in \mathbb{R}^2, \tag{44}$$

with M and q as in (36) and Φ as in (42). Here

$$\ker\{M\} = \{v \in \mathbb{R}^2 : v_2 = -v_1\}.$$

Let $v \in \ker\{M\}$, $v \neq 0$, be given. Then:

$$\langle q, v\rangle + \Phi_\infty(v) = v_2(E_2 - E_1) + \varphi_{PD}(-v_1) + \varphi_{PD}(v_2) = v_2(E_2 - E_1) + 2\varphi_{PD}(v_2).$$

It results that if $v_2 > 0$ then

$$\langle q, v\rangle + \Phi_\infty(v) = v_2(E_2 - E_1) + 2V_1 v_2$$

while if $v_2 < 0$ then

$$\langle q, v\rangle + \Phi_\infty(v) = -v_2(2 \mid V_2 \mid -(E_2 - E_1)) > 0.$$

We may then apply Theorem 1 to assert that the variational inequality $\mathbf{VI}(\mathbf{M}, \mathbf{q}, \Phi, \mathbb{R}^2)$ is stable in the sense of Definition 2.

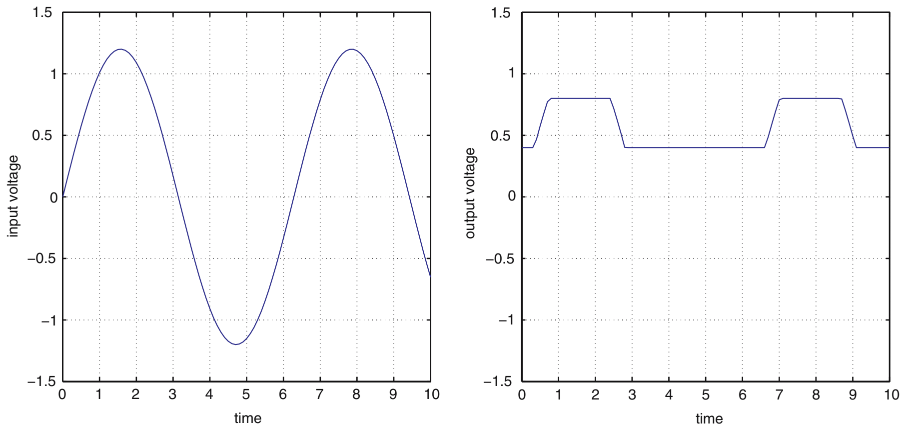
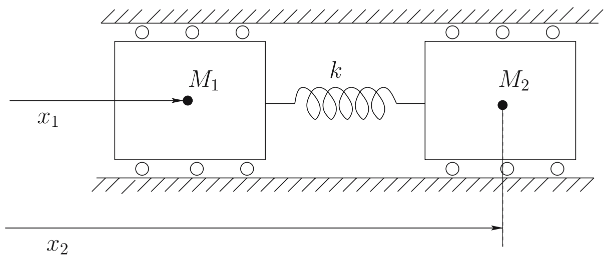


Fig. 7 Double-diode clipper: practical diode

Fig. 8 Two rigid bodies interconnected by a spring



Moreover, the function Φ is strictly convex and the solution Υ^* of (44) is unique and given by:

$$\Upsilon^* = \text{argmin}_{x\in\mathbb{R}^2}\left\{\frac{1}{2}\langle Mx, x\rangle + \langle q, x\rangle + \Phi(x)\right\}. \tag{45}$$

So, for a driven time depending input $t \mapsto u(t)$ the time depending current $t \mapsto i^*(t)$ through the resistor R (Fig. 7) is given by

$$i^*(t) = i_1^*(t) + i_2^*(t) \tag{46}$$

where

$$\left(i_1^*(t)\ i_2^*(t)\right)^T = \text{argmin}_{x\in\mathbb{R}^2}\left\{\frac{1}{2}\langle Mx, x\rangle + (E_1 - u(t))x_1 + (E_2 - u(t))x_2 + \Phi(x)\right\} \tag{47}$$

and the output-signal V_o can then be determined by the formula:

$$V_o(t) = u(t) - Ri^*(t).$$

Example 7 Let us consider the system of two rigid bodies M_1 and M_2 interconnected by a spring of stiffness $k > 0$ and constrained to move only in the horizontal direction. The position of M_1 relative to the origin is represented by x_1 while the position of M_2 relative to the origin is determined by x_2 (Fig. 8).

The mass M_1 is subjected to some external force q_1 and static Coulomb friction force F_1 while the mass M_2 is subjected to some external force q_2 and static Coulomb friction F_2.

The equilibrium states of the system are characterized by the equilibrium equations:

$$F_1 + k(x_2 - x_1) + q_1 = 0$$

and

$$F_2 - k(x_2 - x_1) + q_2 = 0.$$

Static friction Coulomb models for F_1 and F_2 are

$$F_1 \in \begin{cases} -\mu \ \text{ if } \ x_1 < 0 \\ [-\mu, \mu] \ \text{ if } \ x_1 = 0 \\ +\mu \ \text{ if } \ x_1 > 0 \end{cases},$$

and

$$F_2 \in \begin{cases} -\mu \ \text{ if } \ x_2 < 0 \\ [-\mu, \mu] \ \text{ if } \ x_2 = 0 \\ +\mu \ \text{ if } \ x_2 > 0 \end{cases},$$

where $\mu > 0$ denotes Coulomb friction coefficient. These set-valued relations can also be written as

$$F_1 \in -\partial\Psi(x_1)$$

and

$$F_2 \in -\partial\Psi(x_2)$$

where

$$\Psi(x) = \mu|x|, \quad (x \in \mathbf{R}).$$

Setting

$$\Phi(x) = \Psi(x_1) + \Psi(x_2), \quad \forall x = (x_1, x_2) \in \mathbf{R}^2$$

and

$$M = \begin{pmatrix} k & -k \\ k & -k \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{48}$$

we see that our equilibrium system is equivalent to the variational inequality **VI**(**M**, $-$**q**, Φ, $\mathbf{R}^2$), i.e.,

$$X \in \mathbf{R}^2 : \ \langle MX + q, v - X\rangle + \Phi(v) - \Phi(X) \geq 0, \quad \forall v \in \mathbf{R}^2. \tag{49}$$

Here the matrix M is positive semidefinite and symmetric. Moreover

$$\ker\{M\} = \{v \in \mathbb{R}^2 : v_2 = v_1\}.$$

Suppose now that

$$|q_1 + q_2| < 2\mu.$$

For all $v \in \ker\{M\}$, $v \neq (0,0)$ there exists $\alpha \neq 0$ such that $v = (\alpha, \alpha)$ and

$$\Phi_\infty(v) + \langle q, v\rangle = 2\mu|\alpha| + (q_1 + q_2)\alpha. \tag{50}$$

If $\alpha > 0$ then

$$2\mu|\alpha| + (q_1 + q_2)\alpha = (2\mu + (q_1 + q_2))\alpha > 0.$$

If $\alpha < 0$ then

$$2\mu|\alpha| + (q_1 + q_2)\alpha = (2\mu - (q_1 + q_2))|\alpha| > 0.$$

We may thus apply Theorem 1 to assert that our system is stable with respect to data perturbations in the sense of Definition 2.

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